MULTIPLICATIVITY OF THE JLO-CHARACTER

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Abstract. We prove that the Jaffe-Lesniewski-Osterwalder character is compatible with the A_∞-structure of Getzler and Jones.

0. Introduction

Let M be a closed, smooth manifold of even dimension. An order-one, odd, elliptic, pseudodifferential operator D on M gives rise to a K-homology class $[D] \in K_0(M)$. The celebrated Atiyah-Singer index theorem [AS63, AS68a, AS68b] computes the homological Chern character of $[D]$, in terms of the cohomological Chern character of the symbol class of D. There are many proofs known to date. The original one, as explained in [Pal65], proceeds as follows. The *bordsim invariance* and *multiplicativity* of the index map reduces the problem to the computation of the "index genus" $\mathrm{Ind} : \Omega_\bullet^{\mathrm{SO}}(KU) \to \mathbb{Z}$. Then deep results of Thom [Tho54] and Conner-Floyd [CF64] further reduces the problem, in effect, to the Hirzebruch signature theorem and the Bott periodicity theorem. The proof in [AS68a, AS68b] by-passes the bordism computation altogether; here the *strict multiplicativity* (B3$'$) of the index map plays a crucial role.

In this paper, we study the multiplicative property[1] of the index map in noncommutative geometry. In our setting, θ-summable spectral triples play the role of K-homology classes and the JLO character replaces the homological Chern character. There are many other characters, especially if the spectral triple is finitely summable, but it seems that the JLO character is most compatible with the exterior product operation. We show that the JLO character is compatible with the A_∞-exterior product structure on entire chains (Theorem 3.11). The main idea goes back to [GJP91, BG94].

As a corollary, we construct a perturbation of the JLO character that is multiplicative at the chain level (Corollary 3.14). Application to the index theory of transversally elliptic operators will appear elsewhere.

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[1]We do not yet have a good theory of bordism in noncommutative geometry; although preliminary results were announced in Matthias-Moscovici-Pflaum.

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1. Spectral triples

Definition 1.1. A *spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a unital Banach algebra $\mathcal{A}$, a graded Hilbert space $\mathcal{H}$, equipped with a continuous, even representation of $\mathcal{A}$ and a densely defined, self-adjoint, odd operator $\mathcal{D}$ such that

(1) for any $a \in \mathcal{A}$, the commutator $[\mathcal{D}, a] := \mathcal{D}a - a\mathcal{D}$ is bounded, that is, if $\mathrm{dom}(\mathcal{D})$ is the domain of $\mathcal{D}$, then $a \cdot \mathrm{dom}(\mathcal{D}) \subseteq \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, a] : \mathrm{dom}(\mathcal{D}) \to \mathcal{H}$ extends by continuity to a bounded operator on $\mathcal{H}$, and satisfies[2]

$$||a|| + ||[\mathcal{D}, a]|| \leq ||a||_{\mathcal{A}}, \tag{1.1}$$

where $|| \cdot ||$ denotes the operator norm, and

(2) the resolvents $(\mathcal{D} \pm i)^{-1}$ are compact.

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is *θ-summable* if

(3) the operator $e^{-t\mathcal{D}^2}$ is of trace class for any $t > 0$.

Remark 1.2. The compact resolvent condition (2) is equivalent to

(2') the operator $e^{-t\mathcal{D}^2}$ is compact for any (or some) $t > 0$.

Indeed, let

$$\mathcal{C} := \{f \in C_0(\mathbb{R}) \mid f(\mathcal{D}) \text{ is compact}\}. \tag{1.2}$$

Then $\mathcal{C}$ is a closed ideal in $C_0(\mathbb{R})$ and thus $e^{-tx^2} \in C_0(\mathbb{R})$, $t > 0$ belongs to $\mathcal{C}$ iff $\mathcal{C} = C_0(\mathbb{R})$ iff $(x \pm i)^{-1}$ belong to $\mathcal{C}$.

For practical purposes, it is useful to consider essentially self-adjoint operators. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a *pre-spectral triple* if $\mathcal{A}$ is a normed algebra not necessarily complete and $\mathcal{D}$ is required to be just essentially self-adjoint, in Definition 1.1.

Lemma 1.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pre-spectral triple. Then $(\bar{\mathcal{A}}, \mathcal{H}, \bar{\mathcal{D}})$ is a spectral triple, where $\bar{\mathcal{A}}$ denote the completion of $\mathcal{A}$, acting on $\mathcal{H}$ by continuous extension and $\bar{\mathcal{D}}$ denote the closure of $\mathcal{D}$.*

We call $(\bar{\mathcal{A}}, \mathcal{H}, \bar{\mathcal{D}})$ the *closure* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Proof. Let W^1 denote the strong domain of $\bar{\mathcal{D}}$. We show that $\bar{\mathcal{A}}$ preserves W^1. Let $a \in \bar{\mathcal{A}}$ and let $\xi \in W^1$. By definition, there exists a sequence $a_n \in \mathcal{A}$ converging to a and a sequence $\xi_m \in \mathrm{dom}(\mathcal{D})$ converging to ξ such that $\mathcal{D}\xi_m$ converges to some $\eta \in \mathcal{H}$ as $n \to \infty$. Then, by (1.1), the sequence $[\mathcal{D}, a_n]$ is Cauchy and thus have a limit, which we denote $\overline{[\mathcal{D}, a]}$. Again using (1.1), we see that the sequence $a_n\xi_n \in \mathrm{dom}(\mathcal{D})$ converges to $a\xi \in \mathcal{H}$, while $\mathcal{D}(a_n\xi_n)$ converges to $\overline{[\mathcal{D}, a]}\xi + a\eta \in \mathcal{H}$ as $n \to \infty$. Hence $a\xi$ belongs to W^1.

[2] It is enough to require that (1.1) is satisfied up to a multiplicative constant.

It is clear that $[\bar{\mathcal{D}}, a]$ has a bounded extension on $\mathcal{H}$, namely $\overline{[\mathcal{D}, a]}$, which satisfy (1.1), and that $\bar{\mathcal{D}}$ has compact resolvents. □

Example 1.4. Let M be a closed manifold equipped with a smooth measure and let S be a graded, smooth, Hermitian vector bundle over M. Let $\mathcal{H} := L^2(M, S)$ denote the graded Hilbert space of L^2-sections of S. Let $\mathcal{D}$ be an odd, symmetric, elliptic pseudo-differential operator acting on the smooth sections $C^\infty(M, S)$ of S. We consider $\mathcal{D}$ as an unbounded operator on $\mathcal{H}$ with domain $C^\infty(M, S)$. Let $\mathcal{A} := C^\infty(M)$ be the algebra of smooth functions acting on $\mathcal{H}$ by pointwise multiplication, equipped with the norm

$$||a||_{\mathcal{A}} := ||a|| + ||[\mathcal{D}, a]||. \tag{1.3}$$

Then standard ΨDO theory implies that $(\mathcal{A}, \mathcal{H}, \mathcal{D}) = (C^\infty(M), L^2(M, S), \mathcal{D})$ is a pre-spectral triple (cf. [Shu01]).

The following is well-known.

Proposition 1.5. *Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples. Let $\mathcal{A} := \mathcal{A}_1 \otimes_{\mathrm{alg}} A_2$ denote the algebraic tensor product, equipped with the projective tensor product norm $|| \cdot ||_\pi$, and let $\mathcal{H} := \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$ denote the graded Hilbert space tensor product. Let*

$$\mathcal{D} := \mathcal{D}_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} \mathcal{D}_2 \tag{1.4}$$

be the operator with domain

$$\mathrm{dom}(\mathcal{D}) := \mathrm{dom}(\mathcal{D}_1) \widehat{\otimes}_{\mathrm{alg}} \mathrm{dom}(\mathcal{D}_2) \subseteq \mathcal{H}, \tag{1.5}$$

the algebraic graded tensor product. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a pre-spectral triple.

We write $\mathcal{D}_1 \times \mathcal{D}_2$ for the closure of $\mathcal{D}$.

Proof. Suppose $\xi_1 \in \mathcal{H}_1$ and $\xi_2 \in \mathcal{H}_2$ are analytical vectors for $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Then $\xi_1 \widehat{\otimes} \xi_2$ is a smooth vector for $\mathcal{D}$ and, for $t > 0$,

$$\sum_{n=0}^{\infty} \frac{||\mathcal{D}^n(\xi_1 \widehat{\otimes} \xi_2)||}{n!} t^n \leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{k}{n} ||\mathcal{D}_1^k \xi_1|| \, ||\mathcal{D}_2^{n-k} \xi_2|| t^n \tag{1.6}$$

$$= \left(\sum_{k=0}^{\infty} \frac{||\mathcal{D}_1^k \xi_1||}{k!} t^k \right) \left(\sum_{m=0}^{\infty} \frac{||\mathcal{D}_2^m \xi_2||}{m!} t^m \right). \tag{1.7}$$

Hence, choosing $t > 0$ small, we see that $\xi_1 \widehat{\otimes} \xi_2$ is an analytical vector for $\mathcal{D}$. Since the finite linear combination of such elementary tensors is dense in $\mathrm{dom}(\mathcal{D})$, Nelson's analytical vector theorem (cf. [RS75, Theorem X.39]) proves that $\mathcal{D}$ is essentially self-adjoint.

A similar argument shows that

$$e^{-t(\mathcal{D}_1 \times \mathcal{D}_2)^2} = e^{-t\mathcal{D}_1^2} \widehat{\otimes} e^{-t\mathcal{D}_2^2}, \quad t > 0, \tag{1.8}$$

hence, by Remark 1.2, $\mathcal{D}$ has compact resolvents.

Finally, it is clear that $\mathcal{A}$ preserves the domain $\mathrm{dom}(\mathcal{D})$. Moreover, if $a = \sum b_i \otimes c_i$ is an element of $\mathcal{A}$, then

$$[\mathcal{D}, a] = \sum \big([\mathcal{D}_1, b_i]\widehat{\otimes} c_i + b_i \widehat{\otimes}[\mathcal{D}_2, c_i]\big) \tag{1.9}$$

has a bounded extension to $\mathcal{H}$ and the inequality

$$||\sum b_i \otimes c_i|| + ||[\mathcal{D}, \sum b_i \otimes c_i]|| \leq \sum ||b_i|| \cdot ||c_i|| \tag{1.10}$$

$$+ \sum (||[\mathcal{D}_1, b_i]|| \cdot ||c_i|| + ||b_i|| \cdot ||[\mathcal{D}_2, c_i]||) \tag{1.11}$$

$$\leq \sum ||b_i||_{\mathcal{A}_1} \cdot ||c_i||_{\mathcal{A}_2} \tag{1.12}$$

$$\leq ||\sum b_i \otimes c_i||_\pi \tag{1.13}$$

shows that (1.1) is satisfied. □

Definition 1.6. Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be spectral triples. We define their *product* as

$$(\mathcal{A}_1 \otimes_\pi \mathcal{A}_2, \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2, \mathcal{D}_1 \times \mathcal{D}_2), \tag{1.14}$$

where $\otimes_\pi$ denote the projective tensor product.

Remark 1.7. By Proposition 1.5, that the tripe (1.14) is indeed a spectral triple. It follows from equation (1.8) that the product of θ-summable spectral triples is again θ-summable. Finally, note that taking product of spectral triples is *associative* under the natural identifications.

2. The JLO-character

For this section, see [JLO88, Con88, GS89, Con91] for details. Recall that associated to a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, there is an *index map*

$$\mathrm{Ind}_{\mathcal{D}} : K_0(\mathcal{A}) \to \mathbb{Z}, \tag{2.1}$$

given by associating to an idempotent $p \in \mathcal{A} \otimes \mathbb{M}_k$, representing a class in $K_0(\mathcal{A})$, the Fredholm index of the Fredholm operator

$$p\mathcal{D}p : p(\mathcal{H}^0 \otimes \mathbb{C}^k) \to p(\mathcal{H}^1 \otimes \mathbb{C}^k). \tag{2.2}$$

Here $\mathbb{M}_k$ denote the algebra of $k \times k$ complex matrices.

For θ-summable spectral triples, it can be computed "homologically", using the entire cyclic theory of Connes (cf. [Con88]), as follows. We follow the convention of [GS89]. Let

$$C_n(\mathcal{A}) \coloneqq \mathcal{A} \otimes_\pi (\mathcal{A}/\mathbb{C})^{\otimes_\pi n}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{2.3}$$

and let $C_\bullet(\mathcal{A})$ denote the completion of $\bigoplus_{n=0}^\infty C_n(\mathcal{A})$ with respect to the collection of norms

$$|| \oplus_n \alpha_n||_\lambda \coloneqq \sum_{n=0}^\infty \frac{\lambda^n ||\alpha_n||_\pi}{\sqrt{n!}}, \quad \lambda \in \mathbb{Z}_{\geq 1}. \tag{2.4}$$

Let b and B denote the Hochschild and Connes boundary maps on $C_\bullet(\mathcal{A})$ respectively. The *entire cyclic homology* group $HE_\bullet(\mathcal{A})$ is defined as the

homology of the complex $(C_\bullet(\mathcal{A}), b+B)$. The *entire cyclic cohomology* group $HE^\bullet(\mathcal{A})$ is defined using the topological dual $C^\bullet(\mathcal{A})$ of $C_\bullet(\mathcal{A})$.

Notation 2.1. Let

$$\Sigma^n \coloneqq \{t=(t^1,\dots,t^n) \mid 0 \le t^1 \le \cdots \le t^n \le 1\} \subset [0,1]^n \tag{2.5}$$

denote the standard n-simplex equipped with the standard Lebesgue measure $dt = dt^1 \dots dt^n$ with volume $\frac{1}{n!}$.

Definition 2.2 (Jaffe-Lesniewski-Osterwalder [JLO88])**.** Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a θ-summable spectral triple. Denote $\Delta \coloneqq \mathcal{D}^2$ and define $da \coloneqq [\mathcal{D}, a]$, $a \in \mathcal{A}$.

For $(a^0,\dots,a^n) \in \mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^{\otimes n}$ and $t \in \Sigma^n$, we define

$$\langle a^0,\dots,a^n \mid t\rangle_{\mathcal{D}} \coloneqq \mathrm{Str}(a^0 e^{-t^1\Delta} da^1 e^{-(t^2-t^1)\Delta} \dots da^n e^{-(1-t^n)\Delta}), \tag{2.6}$$

where Str is the super-trace on $\mathcal{H}$, and

$$\mathrm{Ch}^n_{\mathcal{D}}(a^0,\dots,a^n) \coloneqq \int_{\Sigma^n} \langle a^0,\dots,a^n \mid t\rangle_{\mathcal{D}}\, dt. \tag{2.7}$$

Then, $\mathrm{Ch}^*_{\mathcal{D}}$ defines an element of $HE^0(\mathcal{A})$ called the *JLO-character* of $(\mathcal{A},\mathcal{H},\mathcal{D})$ and satisfies the abstract index formula

$$\langle \mathrm{Ch}^*_{\mathcal{D}}, \mathrm{Ch}_*(e)\rangle = \mathrm{Ind}_{\mathcal{D}}(e), \quad e \in K_0(\mathcal{A}), \tag{2.8}$$

where $\mathrm{Ch}_*(e) \in HE_0(\mathcal{A})$ denote the entire cyclic homological Chern character of $e \in K_0(\mathcal{A})$. See [GS89] for details.

3. Multiplicativity

The following multiplicative property of the index map is a folklore.

Proposition 3.1. *Let $(\mathcal{A}_1,\mathcal{H}_1,\mathcal{D}_1)$ and $(\mathcal{A}_2,\mathcal{H}_2,\mathcal{D}_2)$ be spectral triples. Then the following diagram is commutative:*

$$\begin{tikzcd} K_0(\mathcal{A}_1)\otimes K_0(\mathcal{A}_2) \arrow[r] \arrow[d, "\mathrm{Ind}_{\mathcal{D}_1}\otimes \mathrm{Ind}_{\mathcal{D}_2}"] & K_0(\mathcal{A}_1\otimes_\pi \mathcal{A}_2) \arrow[d, "\mathrm{Ind}_{\mathcal{D}_1\times\mathcal{D}_2}"] \\ \mathbb{Z}\otimes\mathbb{Z} \arrow[r, equal, "\times"] & \mathbb{Z} \end{tikzcd} \tag{3.1}$$

Proof. Let $e_i \in K_0(\mathcal{A}_i)$, $i \in \{1,2\}$, be given and let $e_1 \otimes e_2$ denote their product in $K_0(\mathcal{A}_1 \otimes_\pi \mathcal{A}_2)$. We need to show that

$$\mathrm{Ind}_{\mathcal{D}_1}(e_1)\cdot \mathrm{Ind}_{\mathcal{D}_2}(e_2) = \mathrm{Ind}_{\mathcal{D}_1\times\mathcal{D}_2}(e_1\otimes e_2). \tag{3.2}$$

As in the proof of [GS89, Theorem D], we may assume that $\mathcal{A}_i$ is an involutive Banach algebra acting involutively on $\mathcal{H}_i$ and e_i is represented by a self-adjoint idempotent $p_i \in \mathcal{A}_i$. Then $(p_i\mathcal{A}_ip_i, p_i\mathcal{H}_i, p_i\mathcal{D}_ip_i)$ is a spectral triple. Moreover we can easily see that

$$p_1\widehat{\otimes}p_2(\mathcal{D}_1\times\mathcal{D}_2)p_1\widehat{\otimes}p_2 = p_1\mathcal{D}_1p_1 \times p_2\mathcal{D}_2p_2 \tag{3.3}$$

on $p_1\widehat{\otimes}p_2\cdot\mathcal{H}_1\widehat{\otimes}\mathcal{H}_2 = p_1\mathcal{H}_1\widehat{\otimes}p_2\mathcal{H}_2$. Hence, we may assume that $p_i = 1 \in \mathcal{A}_i$.

Let P_i denote the projection onto $\ker(\mathcal{D}_i^2)$ and let P denote the projection onto $\ker(\mathcal{D}_1 \times \mathcal{D}_2)^2$. Taking the limit $t \to \infty$ in (1.8), we see that

$$P = P_1 \widehat{\otimes} P_2. \tag{3.4}$$

Hence

$$\operatorname{Ind}(\mathcal{D}_1 \times \mathcal{D}_2) = \operatorname{Str}(P) = \operatorname{Str}(P_1)\operatorname{Str}(P_2) = \operatorname{Ind}(\mathcal{D}_1)\operatorname{Ind}(\mathcal{D}_2). \tag{3.5}$$

The proof is complete. □

Now we study the multiplicative property of the JLO-character. We start by defining an "A_∞-exterior product structure" on entire cyclic chains, following [GJ90, GJP91].

First, the Hochschild shuffle product is defined as follows.

Definition 3.2. Let p, $q \in \mathbb{Z}_{\geq 1}$ be natural numbers and let $S(p,q)$ denote the set $\{(1,1),\dots,(1,p),(2,1),\dots,(2,q)\}$, ordered lexicographically, that is,

$$(1,1) < \cdots < (1,p) < (2,1) < \cdots < (2,q). \tag{3.6}$$

A *permutation* χ of $S(p,q)$is called a (p,q)-*shuffle* if

$$\chi(1,1) < \cdots < \chi(1,p) \quad \text{and} \quad \chi(2,1) < \cdots < \chi(2,q).$$

Let $n = p+q$. Then the ordering (3.6) gives an identification of the set $S(p,q)$ with the set $\{1,\dots,n\}$ and we use this identification to let permutations of $S(p,q)$ act on $\{1,\dots,n\}$.

Let the group of permutations of $\{1,\dots,n\}$ act on $C_n(\mathcal{A})$ by

$$\chi(a^0,a^1,\dots,a^n) \coloneqq (-1)^\chi(a^0,a^{\chi^{-1}(1)},\dots,a^{\chi^{-1}(n)}).$$

Definition 3.3. Let $\alpha = (a^0,a^1,\dots,a^p) \in C_p(\mathcal{A}_1)$ and $\beta = (b^0,b^1,\dots,b^q) \in C_q(\mathcal{A}_2)$ be elementary tensors. The *shuffle product* $\alpha \times \beta \in C_{p+q}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ is defined as

$$\alpha \times \beta \coloneqq \sum_{(p,q)\text{-shuffles}} \chi(a^0 \otimes b^0, a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q).$$

Remark 3.4. We note that the shuffle product extends to

$$C_\bullet(\mathcal{A}_1) \otimes C_\bullet(\mathcal{A}_2) \to C_\bullet(\mathcal{A}_1 \otimes_\pi \mathcal{A}_2).$$

Indeed, if $\alpha \in C_\bullet(\mathcal{A}_1)$ and $\beta \in C_\bullet(\mathcal{A}_2)$, then for any λ, $\mu \in \mathbb{Z}_{\geq 1}$ satisfying $\mu \geq \sqrt{2}\lambda$, we have

$$||\alpha \times \beta||_\lambda = \sum_{n \geq p \geq 0} \frac{\lambda^n ||\alpha_p \times \beta_{n-p}||_\pi}{\sqrt{n!}} \tag{3.7}$$

$$\leq \sum_{n \geq p \geq 0} \frac{\lambda^n \binom{n}{p} ||\alpha_p||_\pi ||\beta_{n-p}||_\pi}{\sqrt{n!}} \tag{3.8}$$

$$\leq ||\alpha||_\mu \cdot ||\beta||_\mu. \tag{3.9}$$

Here we used the fact that the number of $(p,n-p)$-shuffles is $\binom{n}{p} \leq 2^n$.

It is well-known and easy to see that the shuffle product is associative and compatible with the Hochschild boundary map b. However, the Connes boundary map B is *not* a derivation with respect to the shuffle product and we fix this by perturbing the shuffle product by a term called the cyclic shuffle product B_2. The new product we obtain is not associative, but only homotopy associative. In fact, there exists a sequence of operations

$$B_r : C_{p_1}(\mathcal{A}_1) \otimes \cdots \otimes C_{p_r}(\mathcal{A}_r) \to C_{r+p_1+\cdots+p_r}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r) \tag{3.10}$$

such that $B_1 = B$ and $B_2 =$ cyclic shuffle product and B_3 "controls" the failure of associativity et cetera, defined using the following combinatorial device.

Definition 3.5. Let $r \in \mathbb{Z}_{\geq 1}$ and $p_1, \ldots, p_r \in \mathbb{Z}_{\geq 0}$. Let $C(p_1, \ldots, p_r)$ be the set $\{\binom{0}{1}, \ldots, \binom{p_1}{1}, \ldots, \binom{0}{r}, \ldots, \binom{p_r}{r}\}$, ordered lexicographically, that is $\binom{l_1}{k_1} \leq \binom{l_2}{k_2}$ if and only if $k_1 < k_2$ or $k_1 = k_2$ and $l_1 \leq l_2$. This ordering gives an identification of $C(p_1, \ldots, p_r)$ with the set $\{1, \ldots, r + p_1 + \ldots p_r\}$.

A $(p_1, \ldots, p_r)$-cyclic shuffle is a permutation σ of the set $C(p_1, \ldots, p_r)$ such that

(1) $\sigma\binom{0}{i_1} < \sigma\binom{0}{i_2}$ if $i_1 < i_2$ and
(2) for each $1 \leq i \leq r$, there is a number $0 \leq j_i \leq p_i$ such that

$$\sigma\binom{j_i}{i} < \cdots < \sigma\binom{p_i}{i} < \sigma\binom{0}{i} < \cdots < \sigma\binom{j_i - 1}{i}.$$

Definition 3.6. For $\alpha_i = (a_1^0, \ldots, a_1^{p_i}) \in C_{p_i}(\mathcal{A}_i)$, we define $B_r(\alpha_1, \ldots, \alpha_r) \in C_{r+p_1+\cdots+p_r}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r)$ as

$$B_r(\alpha_1, \ldots, \alpha_r) \coloneqq \sum_{\sigma} \sigma(1, a_1^0, \ldots, a_1^{p_1}, \ldots, a_r^0, \ldots, a_r^{p_r}), \tag{3.11}$$

where $a_i^j \in \mathcal{A}_i$ is considered an element of $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r$ via $a_i \mapsto 1_{\mathcal{A}_1} \otimes \cdots \otimes 1_{\mathcal{A}_{i-1}} \otimes a_i \otimes 1_{\mathcal{A}_{i+1}} \otimes \cdots \otimes 1_{\mathcal{A}_r}$ and the summation is over all $(p_1, \ldots, p_r)$-cyclic shuffles. For $\alpha \in C_\bullet(\mathcal{A}_1)$ and $\beta \in C_\bullet(\mathcal{A}_2)$, we also write

$$\alpha \times' \beta \coloneqq B_2(\alpha, \beta) \tag{3.12}$$

and call $\times'$ the *cyclic shuffle product.*

Remark 3.7. If $\mathcal{A}$ is commutative, we recover the B_k terms of the A_∞-structure of Getzler and Jones [GJ90] using the multiplication map.

One motivation for the definition of the shuffles and the cyclic shuffles is the following.

We let permutations on $\{1, \ldots, n\}$ act on $[0, 1]^n$ by

$$\chi(t^1, \ldots, t^n) \coloneqq (t^{\chi^{-1}(1)}, \ldots, t^{\chi^{-1}(n)}). \tag{3.13}$$

Then for any element $t \in [0, 1]^n$, such that all the entries are distinct, there exists a *unique* permutation χ such that $\chi(t)$ is an element of Σ^n, *i.e.* entries of $\chi(t)$ are in increasing order. Therefore, permutations give a

decomposition of $[0,1]^n$ into n-simplices. The shuffles and the cyclic shuffles give decompositions of product simplices.

Lemma 3.8 (Getzler-Jones-Petrack [GJP91])**.** *(1) Let p and q be natural numbers. For a (p,q)-shuffle χ, define*

$$\Sigma(\chi) \coloneqq \{(s,t) \in \Sigma^p \times \Sigma^q \subset [0,1]^{p+q} \mid \chi(s,t) \in \Sigma^{p+q}\}. \tag{3.14}$$

Then, $\Sigma(\chi)$ is a $(p+q)$-simplex and, up to a set of measure zero, $\Sigma^p \times \Sigma^q$ is the disjoint union of the sets $\Sigma(\chi)$, χ shuffle.

(2) Similarly, for a cyclic shuffle σ, define

$$\Sigma(\sigma) \coloneqq \{(s,t_1,\dots,t_r) \mid \sigma(s+t_1+\cdots+t_r) \in \Sigma^{r+p_1+\cdots+p_r}\}, \tag{3.15}$$

where $(s,t_1,\dots,t_r) \in \Sigma^r \times \Sigma^{p_1} \times \cdots \times \Sigma^{p_r} \subset [0,1]^{r+p_1+\cdots+p_r}$ and $s+t^1+\cdots+t^r$ denote the $(r+p_1+\cdots+p_r)$-tuple

$$(s^1, s^1+t_1^1, \dots s^1+t_1^{p_1}, \dots, s^r, s^r+t_r^1, \dots, s^r+t_r^{p_r}) \tag{3.16}$$

considered modulo 1.

Then, $\Sigma(\sigma)$ is a $(r+p_1+\cdots+p_r)$-simplex and, up to a set of measure zero, $\Sigma^r \times \Sigma^{p_1} \times \cdots \times \Sigma^{p_r}$ is the disjoint union of the sets $\Sigma(\sigma)$, σ cyclic shuffle.

□

Remark 3.9. It follows from Lemma 3.8(2) that the number of $(p_1,\dots,p_r)$-cyclic shuffles is

$$\binom{r+p_1+\cdots+p_r}{r,p_1,\dots,p_r} = \frac{(r+p_1+\cdots+p_r)!}{r!p_1!\dots p_r!}. \tag{3.17}$$

An argument similar to Remark 3.4, proves that the operation B_r extend to

$$B_r : C_\bullet(\mathcal{A}_1) \otimes_\pi \cdots \otimes_\pi C_\bullet(\mathcal{A}_r) \to C_{r+\bullet}(\mathcal{A}_1 \otimes_\pi \cdots \otimes_\pi \mathcal{A}_r). \tag{3.18}$$

For $B = B_1$, we have the following. First note that the expression (2.6) makes sense for a^0 in $\mathcal{A} + [\mathcal{D},\mathcal{A}]$. We write

$$\langle d\alpha \mid t\rangle_{\mathcal{D}} \coloneqq \left\langle da^0, a^1, \dots, a^n \mid t\right\rangle_{\mathcal{D}} \tag{3.19}$$

for $\alpha = (a^0,\dots,a^n) \in C_n(\mathcal{A})$ and $t \in \Sigma^n$.

Proposition 3.10 ([GS89, Lemma 2.2(2)])**.** *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a θ-summable spectral triple. Then*

$$B\mathrm{Ch}_{\mathcal{D}}(\alpha) = \int_{\Sigma^n} \langle d\alpha \mid t\rangle\, dt, \tag{3.20}$$

for $\alpha \in C_n(\mathcal{A})$. □

Following is the analogue of [GJP91, Proposition 4.1, 4.2] and [BG94, Theorem 3.2]. Part (2) below uses Proposition 3.10 and can be considered an extension of it to the case $r \geq 2$.

Theorem 3.11. *(1) Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be θ-summable spectral triples. Then*

$$\mathrm{Ch}_{\mathcal{D}_1\times\mathcal{D}_2}(\alpha_1 \times \alpha_2) = \mathrm{Ch}_{\mathcal{D}_1}(\alpha_1)\mathrm{Ch}_{\mathcal{D}_2}(\alpha_2), \tag{3.21}$$

for $\alpha_1 \in C_\bullet(\mathcal{A}_1)$ and $\alpha_2 \in C_\bullet(\mathcal{A}_2)$.

(2) Let $(\mathcal{A}_i, \mathcal{H}_i, \mathcal{D}_i)$, $1 \le i \le r$, be θ-summable spectral triples. Then

$$\mathrm{Ch}_{\mathcal{D}_1\times\mathcal{D}_2\times\cdots\times\mathcal{D}_r} B_r(\alpha_1, \dots, \alpha_r) = \frac{1}{r!} B\mathrm{Ch}_{\mathcal{D}_1}(\alpha_1)\dots B\mathrm{Ch}_{\mathcal{D}_r}(\alpha_r), \tag{3.22}$$

for $\alpha_i \in C_\bullet(\mathcal{A}_i)$, $1 \le i \le r$. In particular, for $r = 2$,

$$\mathrm{Ch}_{\mathcal{D}_1\times\mathcal{D}_2}(\alpha_1 \times' \alpha_2) = \frac{1}{2} B\mathrm{Ch}_{\mathcal{D}_1}(\alpha) B\mathrm{Ch}_{\mathcal{D}_2}(\alpha_2). \tag{3.23}$$

Note that Remark 1.7 shows that the theorem is well-posed.

Proof. First we prove (1). Let $\alpha_1 = (a^0, a^1, \dots, a^p) \in C_p(\mathcal{A}_1)$ and $\alpha_2 = (b^0, b^1, \dots, b^q) \in C_q(\mathcal{A}_2)$ and let $\gamma = (c^{0,0}, c^{1,1}, \dots, c^{1,p}, c^{2,1}, \dots, c^{2,q}) \in C_{p+q}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ denote the element

$$(a^0 \otimes b^0, a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q). \tag{3.24}$$

Then using (1.8) and (1.9), we see that

$$e^{-t(\mathcal{D}_1\times\mathcal{D}_2)^2}[\mathcal{D}_1 \times \mathcal{D}_2, c^{i,j}] = \begin{cases} e^{-t\Delta_1}[\mathcal{D}_1, a^j]\widehat{\otimes} e^{-t\Delta_2}, & i = 1 \\ e^{-t\Delta_1}\widehat{\otimes} e^{-t\Delta_2}[\mathcal{D}_2, b^j], & i = 2 \end{cases}, \tag{3.25}$$

for $t > 0$. Now it is straightforward to check that for any (p, q)-shuffle χ and $(s, t) \in \Sigma(\chi) \subset \Sigma^p \times \Sigma^q$ with $u = \chi(s, t) \in \Sigma^{p+q}$,

$$\langle \chi(\gamma) \mid u \rangle_{\mathcal{D}_1\times\mathcal{D}_2} = \langle \alpha_1 \mid s \rangle_{\mathcal{D}_1} \cdot \langle \alpha_2 \mid t \rangle_{\mathcal{D}_2}. \tag{3.26}$$

Integrating over $\Sigma(\chi)$ and summing over all the (p, q)-shuffles χ, we get the result, using Lemma 3.8(1).

The proof of (2) is similar. Let γ denote the element

$$(1, a_1^0, \dots, a_1^{p_1}, \dots, a_r^0, \dots, a_r^{p_r}) \tag{3.27}$$

in $C_{r+p_1+\cdots+p_r}(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_r)$. Then for any cyclic $(p_1, \dots, p_r)$-shuffle σ, and $(s, t_1, \dots, t_r) \in \Sigma(\sigma) \subset \Sigma^r \times \Sigma^{p_1} \times \cdots \times \Sigma^{p_r}$ with $v = \sigma(s + t_1 + \cdots + t_r) \in \Sigma^{r+p_1+\cdots+p_r}$, we have

$$\langle \sigma(\gamma) \mid v \rangle_{\mathcal{D}_1\times\cdots\times\mathcal{D}_r} = \langle d\alpha_1 \mid t_1 \rangle_{\mathcal{D}_1} \cdot \cdots \cdot \langle d\alpha_r \mid t_r \rangle_{\mathcal{D}_r}.$$

Now Lemma 3.8(2) and Proposition 3.10 completes the proof. The factor $1/r!$ in (3.22) is the volume of Σ^r. □

Definition 3.12. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a θ-summable spectral triple. We define the *perturbed JLO cocycle* as

$$\mathrm{Ch}^{\mathrm{pert}}_\bullet = \mathrm{Ch}_\bullet + \frac{1}{\sqrt{2}} B\mathrm{Ch}_{\bullet-1} \tag{3.28}$$

Remark 3.13. (1) The perturbed JLO character is a cocycle:

$$\begin{aligned}(b+B)\mathrm{Ch}^{\mathrm{pert}}_{\bullet} &= (b+B)(\mathrm{Ch}_{\bullet} + 2^{-\frac{1}{2}} B\mathrm{Ch}_{\bullet-1}) \\ &= 2^{-\frac{1}{2}} bB\mathrm{Ch}_{\bullet-1} \\ &= -2^{-\frac{1}{2}} Bb\mathrm{Ch}_{\bullet-1} \\ &= -2^{-\frac{1}{2}} B(b+B)\mathrm{Ch}_{\bullet-1} \\ &= 0.\end{aligned}$$

(2) The perturbed JLO character has mixed parity: $\mathrm{Ch}_{\bullet}$ is even and $B\mathrm{Ch}_{\bullet_{-1}}$ is odd. The pairing with K_0 depends only on the even part, hence Ch and $\mathrm{Ch}^{\mathrm{pert}}$ have the same pairing with $K_0(\mathcal{A})$.

Corollary 3.14. *Let $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ be θ-summable spectral triples. Then for $\alpha \in C_{\bullet}(\mathcal{A}_1)$ and $\beta \in C_{\bullet}(\mathcal{A}_2)$*

$$\mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha \times \beta + \alpha \times' \beta) = \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1}(\alpha) \cdot \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_2}(\beta). \tag{3.29}$$

Proof. For a θ-summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we write

$$\delta(a^0, a^1 \dots, a^p) := \frac{1}{\sqrt{2}}([\mathcal{D}, a^0], a^1, \dots, a^p). \tag{3.30}$$

Note that $[\mathcal{D}, a^0]$ does *not* necessarily belong to $\mathcal{A}$, but this causes no trouble.

Then we can write

$$\mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}} = \mathrm{Ch}_{\mathcal{D}} \circ (1+\delta). \tag{3.31}$$

Now we write δ_1, δ_2 and δ_{12} for the δ corresponding to the spectral triples $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1)$, $(\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$ and $(\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1) \times (\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)$, respectively. Then, using (1.9) we see that

$$\delta_{12}(\alpha \times \beta) = \delta_1(\alpha) \times \beta + \alpha \times \delta_2(\beta). \tag{3.32}$$

Moreover $\delta_{12}(\alpha \times' \beta) = 0$ because all the summands start with the term $[\mathcal{D}_1 \times \mathcal{D}_2, 1 \otimes 1] = 0$.

Therefore,

$$\begin{aligned}&\mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha \times \beta + \alpha \times' \beta) \\ &\quad = \mathrm{Ch}_{\mathcal{D}_1 \times \mathcal{D}_2}\left((1+\delta_{12})(\alpha \times \beta + \alpha \times' \beta)\right) \\ &\quad = \mathrm{Ch}_{\mathcal{D}_1 \times \mathcal{D}_2}(\alpha \times \beta + \alpha \times' \beta + \delta_1(\alpha) \times \beta + \alpha \times \delta_2(\beta)) \\ &\quad = \mathrm{Ch}_{\mathcal{D}_1}(\alpha)\mathrm{Ch}_{\mathcal{D}_2}(\beta) + \mathrm{Ch}_{\mathcal{D}_1}(\delta_1\alpha)\mathrm{Ch}_{\mathcal{D}_2}(\delta_2\beta) \\ &\qquad + \mathrm{Ch}_{\mathcal{D}_1}(\delta_1\alpha)\mathrm{Ch}_{\mathcal{D}_2}(\beta) + \mathrm{Ch}_{\mathcal{D}_1}(\alpha)\mathrm{Ch}_{\mathcal{D}_2}(\delta_2\beta) \\ &\quad = \mathrm{Ch}_{\mathcal{D}_1}\left((1+\delta_1)(\alpha)\right)\mathrm{Ch}_{\mathcal{D}_2}\left((1+\delta_2)(\beta)\right) \\ &\quad = \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_1}(\alpha) \cdot \mathrm{Ch}^{\mathrm{pert}}_{\mathcal{D}_2}(\beta),\end{aligned}$$

by Theorem 3.11. □

Corollary 3.15. *For theta-summable spectral triples, the perturbed JLO character implements the diagram in Lemma 3.1.* □

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